Mini-course on GAP – Lecture 4

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Finite simple groups

Theorem: classification of finite simple groups

Every finite simple group is isomorphic to one of the following groups:

- a member of one of three infinite classes of such, namely:
  - the cyclic groups of prime order,
  - the alternating groups of degree at least 5,
  - the groups of Lie type
- one of 26 groups called the "sporadic groups"
- the Tits group (which is sometimes considered the 27th sporadic group).

Jacques Tits

J. Tits, 1930 – 2021

- Groups and geometries
- Buildings
- Generalized Polygons
- Abel prize 2008 together with John Thompson

Geometries from groups and vice versa

Question

Consider $\mathrm{PSO}^-(4, q) \leqslant \mathrm{PSL}(4, q)$. The group $\mathrm{PSL}(4, q)$ acts naturally on the elements of $\mathrm{PG}(3, q)$. The group $\mathrm{PSO}^-(4, q)$ acts naturally on the elements of an *elliptic quadric*. Now consider $\mathrm{Sz}(2^{2e+1}) \leqslant \mathrm{PSL}(4, 2^{2e+1})$. Is there a similar object where the Suzuki group acts naturally on?

Historical note

Suzuki discovered the infinite family group (1960), Ree realized that it was a subgroup of $\mathrm{PSL}(4, q)$ and even of $\mathrm{PSp}(4, q)$.

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Finite projective spaces

definition

Let $K = \mathrm{GF}(q)$ be a finite field. The projective space $\mathrm{PG}(d, q)$ is the incidence geometry that consists of all subspaces of the $d+1$-dimensional vectors space $V(d+1, q)$.

- *projective points* are vector lines,
- *projective lines* are vector planes,
- ...
- *projective hyperplanes* are vector hyperplane.

We can consider this as an *incidence structure*, the elements have a *type*, e.g. there projective dimension. The incidence is a symmetric relation, which is symmetrized containment, and elements of the same type are not incident (unless they are equal). Elements are represented by an underlying subspace of the vector space, there are some obvious possibilities such as span and meet.

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Automorphisms

definition

A *collineation* of an incidence geometry is a type and incidence preserving bijection of the geometry.

Elements of the matrix group $\mathbf{GL}(d+1,q)$ *induce* collineations of $\mathrm{PG}(d,q)$. We can extend to semi-linear maps of the underlying vector space. This group is denoted by $\Gamma\mathrm{L}(d+1,q)$

Fundamental theorem of projective geometry

Every collineation of $\mathrm{PG}(d,q)$ is induced by an element of $\mathbf{GL}(d+1,q)$. The only elements of $\mathbf{GL}(d+1,q)$ fixing all elements of $\mathrm{PG}(d,q)$ are the scalar matrices.

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Projective groups

- $\mathrm{P\Gamma L}(d+1,q) = \mathrm{\Gamma L}(d+1,q)/\mathrm{Sc}(d+1,q)$
- $\mathrm{PGL}(d+1,q) = \mathrm{GL}(d+1,q)/\mathrm{Sc}(d+1,q)$
- $\mathrm{PSL}(d+1,q) = \mathrm{SL}(d+1,q)/\mathrm{Sc}(d+1,q)$. This is a simple group!

Beniamino Segre

B. Segre, 1903 – 1977

- Classical algebraic geometry
- Arcs and caps in projective spaces
- Combinatorial approach to algebraic curves and surfaces.

Arcs and caps

A conic in $\mathrm{PG}(2, q)$ is a non-degenerate curve of degree 2. It is an example of an *arc*, i.e. a set of points such that no three of them are collinear. *Ovals* are arcs of precisely $q+1$ points. When q is even, there exist *hyperovals*, i.e. an arc of size $q+2$.
Important question studied by Segre: do there exist ovals different from conics?

An elliptic quadric in $\mathrm{PG}(3, q)$ is a hypersurface of degree 2 not containing lines. It is an example of a *cap*, i.e. a set of points meeting all lines in 0, 1 or 2 points. An *ovoid* is a cap of size q^2+1.
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Simple groups

- The setwise stabilizers of quadrics in the special linear group will produce simple groups.
- These groups are often referred to as the *orthogonal groups*.
- Refer to e.g. P. Kleidman and M. Liebeck, *The Subgroup Structure of the Finite Classical Groups*.

Back to ovoids

- Elliptic quadrics are *classical* ovoids in $\mathrm{PG}(3, q)$.
- Segre looked for non-classical examples as a combinatorial research problem. This ignited a whole school of Italian finite geometers working on related questions.
- Tits looked for a geometrical object in $\mathrm{PG}(3, q)$ to interpret the Suzuki groups. It turned out that both Segre and Tits were looking for the same object!

Generalized quadrangles

definition

A *finite generalized quadrangle* is a point-line geometry such that

- for each point P there are exactly $t+1$ lines incident with P, for a fixed $t \geq 1$;
- for each line l there are exactly $s+1$ points incident with l;
- for each line l and each point P not incident with l, there exists a unique line m through P meeting l in a unique point.

The pair (s, t) is called the *order* of the GQ.

Generalized quadrangles

definition

A finite generalized quadrangle is a point-line geometry such that its incidence graph is bipartite and has diameter 4 and girth 8.

The symplectic quadranlge (1)

- Consider a symplectic polarity of $\mathrm{PG}(3, q)$.
- The point-line geometry consisting of the absolute points and absolute lines is a GQ of order (q, q)
- Check this in GAP using a graph.
- Observe that for q even, the automorphism group of the graph is twice as large as the setwise stabilizer group of the lines.

The symplectic quadrangle (2)

- Consider an elliptic quadric $\mathrm{Q}^-(3,q)$ in $\mathrm{PG}(3,q)$, q even.
- Let $\mathcal{L}$ be the set of lines tangent to $\mathrm{Q}^-(3,q)$, and $\mathcal{P}$ be the set of points of $\mathrm{PG}(3,q)$.
- Check that the point-line geometry with point set $\mathcal{P}$, $\mathcal{L}$ line set $\mathcal{L}$ and the natural incidence is a GQ!

Conclusion: we observed two ways of constructing the same GQ!.

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The symplectic quadrangle $\mathrm{W}(3, q)$

- J. Tits proved that $\mathrm{W}(3, q)$ is *self polar* if and only if $q = 2^{2e+1}$.
- What is the set of absolute points with relation to such a polarity?
- We can try to find out using GAP !

Axiomatic planes

definition

A *projective plane* is a point-line geometry such that

- every two points determine exactly one line;
- every two lines determine exactly one point;
- there are four points of which no three are collinear.

Classical means coordinatized by a field. Also in the finite case, there is a big interest in non-Desarguesian projective planes. We are going to investigate a generic way to construct projective planes. Note that the incidence graph of a projective plane is bipartite, and of diameter 3 and girth 6.

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Spreads of projective spaces

definition

Consider $\mathrm{PG}(3, q)$. A *spread* is a set of lines partitioning the point set. It has necessarily $q^2 + 1$ lines.

It is possible to construct a spread of $\mathrm{PG}(3, q)$ using field reduction:

- Consider the additive group $V(4, q), +$ this is isomorphic with the additive group $\mathrm{GF}(q^4), +$, which is isomorphic with $V(2, q^2), +$.
- $V(2, q)$ is the underlying vector space of the projective line $\mathrm{PG}(1, q^2)$. Each point is represented by a vector line of $V(2, q^2)$, of which there are exactly $q^2 + 1$.
- A vector line of $V(2, q^2)$ is an additive group isomorphic with $\mathrm{GF}(q^2), +$, which is isomorphic with $V(2, q), +$. So a vector line of $V(2, q^2)$ becomes a vector plane of $V(4, q)$.
- Hence the points of the projective line $\mathrm{PG}(1, q^2)$ become lines of $\mathrm{PG}(3, q)$.

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Reguli

definition

A *regulus* in $\mathrm{PG}(3,q)$ is a set $\mathcal{R}$ of lines such that

- $|\mathcal{R}| = q+1$
- If $l_1, l_2 \in \mathcal{R}$ and $l_1 \neq l_2$ then $l_1 \cap l_2 = \emptyset$
- If a line l meets three distinct lines of $\mathcal{R}$, then it meet all lines of $\mathcal{R}$.

Theorem

Let l_1, l_2, l_3 be three mutually skew lines of $\mathrm{PG}(3,q)$, then there exists a unique regulus of $\mathrm{PG}(3,q)$ containing these lines.

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Regular spreads

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A line spread of $\mathrm{PG}(3, q)$ is regular if every regulus determined any three elements of it is contained in it.

Non-classical projective planes: André-Bruck-Bose

The so-called André-Bruck-Bose construction is a way to construct projective planes from spreads. When the spread is regular, the projective planes turns out to be Desarguesian. Indeed, any non-regular spread gives rise to non-Desarguesian projective planes!

André-Bruck-Bose

Consider a line spread $\mathcal{S}$ in $\mathrm{PG}(3, q)$. Embed $\mathrm{PG}(3, q)$ as a hyperplane π_∞ in $\mathrm{PG}(4, q)$. Now define a point-line geometry $\Pi_\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ as follows.

The elements of $\mathcal{P}$ are

(i) the points of $\mathrm{PG}(4, q) \setminus \pi_\infty$;

(ii) the elements of $\mathcal{S}$.

The elements of $\mathcal{L}$ are

(a) the planes of $\mathrm{PG}(2t + 2, q)$ meeting π_∞ in an element of $\mathcal{S}$;

(b) the hyperplane π_∞.

Demos/exercises with GAP

- Construct a regular spread of $\mathrm{PG}(3, q)$ using field reduction. You may use e.g. NaturalEmbeddingByFieldReduction
- Write a function to swap a regulus with its opposite regulus. You can check that the obtained spread is non-isomorphic with a regular spread by computing its stabilizer group.
- Construct the projective plane. One possibility is to construct it as a bipartite graph.